# Albanese varieties with modulus and Hodge theory

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## 1 Introduction

**1.1.** Let $X$ be a proper smooth variety over a field $k$ of characteristic 0, and let $\mathrm{Alb}(X)$ be the Albanese variety of $X$. In the work [Ru], the second author constructed generalized Albanese varieties $\mathrm{Alb}_{\mathcal{F}}(X)$, which are commutative connected algebraic groups over $k$ with surjective homomorphisms $\mathrm{Alb}_{\mathcal{F}}(X) \to \mathrm{Alb}(X)$ (see section 5 for a review). If $Y$ is an effective divisor on $X$, a special case of $\mathrm{Alb}_{\mathcal{F}}(X)$ becomes the generalized Albanese variety $\mathrm{Alb}(X, Y)$ of $X$ of modulus $Y$ (cf. section 5). This is a higher dimensional analogue of the generalized Jacobian variety with modulus of Rosenlicht-Serre. Note that the divisor $Y$ can have multiplicity, and so the algebraic group $\mathrm{Alb}(X, Y)$ can have an additive part.

Assume now $k = \mathbb{C}$. The purpose of this paper is to give Hodge theoretic presentations (Thm. 1.4) of $\mathrm{Alb}(X, Y)$.

The case $Y$ has no multiplicity was studied in the work [BaS] of Barbieri-Viale and Srinivas. A Hodge theoretic presentation of a generalized Albanese variety in the case without modulus but allowing the singularity of $X$ was given in the work [ESV] of Esnault, Srinivas and Viehweg.

**1.2.** First we review the curve case. Let $X$ be a proper smooth curve over $\mathbb{C}$ and let $Y$ be an effective divisor on $X$. In this case, the Albanese variety $\mathrm{Alb}(X, Y)$ of $X$ relative to $Y$ coincides with the generalized Jacobian variety $J(X, Y)$ of $X$ relative to $Y$. In the following, we will write the complex analytic space associated to $X$ simply by $X$, and the sheaf of holomorphic functions on it by $\mathcal{O}_X$. Let $I = \mathrm{Ker}(\mathcal{O}_X \to \mathcal{O}_Y)$ be the ideal of $\mathcal{O}_X$ which defines $Y$. The cohomology below is for the topology of the analytic space $X$ (not for Zariski topology).

The generalized Jacobian variety $J(X, Y)$ is the kernel of the degree map $\mathrm{Pic}(X, Y) \to \mathbb{Z}$ where $\mathrm{Pic}(X, Y) = H^1(X, \mathrm{Ker}(\mathcal{O}_X^\times \to \mathcal{O}_Y^\times))$. Let $j : X - Y \longrightarrow X$ be the inclusion map and let $j_!\mathbb{Z}(1)$ be the 0-extension of the constant sheaf $\mathbb{Z}(1)$ of $X - Y$ to $X$. (For $r \in \mathbb{Z}$, $\mathbb{Z}(r)$ denotes $\mathbb{Z}(2\pi i)^r$ as usual.) Then we have an exact sequence

$$0 \longrightarrow j_!\mathbb{Z}(1) \longrightarrow I \xrightarrow{\exp} \mathrm{Ker}(\mathcal{O}_X^\times \to \mathcal{O}_Y^\times) \longrightarrow 0$$

and hence we have an isomorphism

$$\mathrm{Pic}(X, Y) \cong H^2(X, [j_!\mathbb{Z}(1) \to I]). \qquad (1)$$

Here in the complex $[j_!\mathbb{Z}(1) \to I]$, $j_!\mathbb{Z}(1)$ is put in degree 0.

We have another presentation of $J(X,Y)$ given in (2) below. Let $I_1$ be the ideal of $\mathcal{O}_X$ which defines the reduced part of $Y$ and let $J = II_1^{-1} \subset \mathcal{O}_X$. Note that the composition of the two inclusion maps of complexes

$$[I \xrightarrow{d} J\Omega^1_X] \longrightarrow [I \xrightarrow{d} \Omega^1_X] \longrightarrow [I_1 \xrightarrow{d} \Omega^1_X]$$

is a quasi-isomorphism. Hence we have an isomorphism in the derived category

$$[I \xrightarrow{d} \Omega^1_X] \cong [I_1 \xrightarrow{d} \Omega^1_X] \oplus (\Omega^1_X/J\Omega^1_X)[-1].$$

Since $j_!\mathbb{C} \longrightarrow [I_1 \xrightarrow{d} \Omega^1_X]$ is a quasi-isomorphism, we have an exact sequence

$$(2) \qquad H^0(X,\Omega^1_X) \longrightarrow H^1_c(X-Y,\mathbb{C}/\mathbb{Z}(1)) \oplus H^0(X,\Omega^1_X/J\Omega^1_X) \longrightarrow J(X,Y) \longrightarrow 0.$$

(Here $H_c$ is the cohomology with compact supports.)

**1.3.** Now let $X$ be a proper smooth variety over $\mathbb{C}$ of dimension $n$ and let $Y$ be an effective divisor on $X$.

Again in the following theorem, cohomology groups are for the topology of the complex analytic spaces, and the notation $\mathcal{O}$ and $\Omega$ stand for analytic sheaves.

Let $I$ be the ideal of $\mathcal{O}_X$ which defines $Y$, let $I_1$ be the ideal of $\mathcal{O}_X$ which defines the reduced part of $Y$, and let $J = II_1^{-1} \subset \mathcal{O}_X$.

**Theorem 1.4.** (1) *We have an exact sequence*

$$0 \longrightarrow \mathrm{Alb}(X,Y) \longrightarrow H^{2n}(X,\mathcal{D}_{X,Y}(n)) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0,$$

*where for $r \in \mathbb{Z}$, $\mathcal{D}_{X,Y}(r)$ denotes the kernel of the surjective homomorphism of complexes $\mathcal{D}_X(r) \to \mathcal{D}_Y(r)$ with $D_X(r)$ the Deligne complex*

$$[\mathbb{Z}(r) \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \dots \xrightarrow{d} \Omega^{r-1}_X]$$

*and $\mathcal{D}_Y(r)$ the similar complex*

$$[\mathbb{Z}(r)_Y \to \mathcal{O}_Y \xrightarrow{d} \Omega^1_Y \xrightarrow{d} \dots \xrightarrow{d} \Omega^{r-1}_Y].$$

(2) *We have an exact sequence*

$$H^{n-1}(X,\Omega^n_X) \longrightarrow H^{2n-1}_c(X-Y,\mathbb{C}/\mathbb{Z}(n)) \oplus H^{n-1}(X,\Omega^n_X/J\Omega^n_X) \longrightarrow \mathrm{Alb}(X,Y) \longrightarrow 0.$$

Note that the case $n=1$ of Thm. 1.4 (1) (resp. (2)) becomes the presentation of $J(X,Y)$ given by (1) (resp. (2)) in 1.2.

**Remark 1.5.** We give some remarks on this theorem.

(a) The case $Y = 0$ of Thm. 1.4 (1) is nothing but the well known exact sequence

$$0 \longrightarrow \mathrm{Alb}(X) \longrightarrow H^{2n}(X, \mathcal{D}_X(n)) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0 \tag{3}$$

by using the Deligne cohomology $H^{2n}(X, \mathcal{D}_X(n))$. (Usually the Deligne cohomology $H^m(X, \mathcal{D}_X(r))$ is denoted by $H^m_D(X, \mathbb{Z}(r))$.)

The case $Y = 0$ of Thm. 1.4 (2) is nothing but the usual presentation

$$\mathrm{Alb}(X) \cong H_{\mathbb{Z}} \backslash H_{\mathbb{C}} / F^0 H_{\mathbb{C}} \tag{4}$$

of the Albanese variety $\mathrm{Alb}(X)$ of $X$, where $(H_{\mathbb{Z}}, H_{\mathbb{C}}, F^\bullet)$ is the following Hodge structure of weight $-1$. $H_{\mathbb{Z}} = H^{2n-1}(X, \mathbb{Z}(n))/(\text{torsion part})$, $H_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} H_{\mathbb{Z}} = H^{2n-1}(X, \Omega^\bullet_X)$, and $F^\bullet$ is the Hodge filtration on $H_{\mathbb{C}}$ defined as

$$F^{-1} = H_{\mathbb{C}}, \quad F^0 = H^{n-1}(X, \Omega^n_X), \quad F^1 = 0.$$

(b) Recall that the presentations (3) and (4) of $\mathrm{Alb}(X)$ are related as follows. Consider the exact sequence of complexes $0 \to \Omega^{\leq n-1}_X[-1] \to \mathcal{D}_X(n) \to \mathbb{Z}(n) \to 0$, where $\Omega^{\leq n-1}_X$ denotes the part of degree $\leq n-1$ of the de Rham complex $\Omega^\bullet_X$, which is actually a quotient complex of $\Omega^\bullet_X$. By taking the cohomology associated to this exact sequence, we have an exact sequence

$$H^{2n-1}(X, \mathbb{Z}(n)) \longrightarrow H^{2n-1}(X, \Omega^{\leq n-1}_X) \longrightarrow H^{2n}_D(X, \mathbb{Z}(n)) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0.$$

Since

$$H^{2n-1}(X, \Omega^{\leq n-1}_X) \cong H^{2n-1}(X, \Omega^\bullet_X)/H^{n-1}(X, \Omega^n_X) \cong H^{2n-1}(X, \mathbb{C})/H^{n-1}(X, \Omega^n_X),$$

the exact sequence (4) is equivalent to (3).

(c) (1) and (2) of Thm. 1.4 are related similarly. Let $S$ be the subcomplex of the de Rham complex $\Omega^\bullet_X$ of $X$ defined by $S^p = \mathrm{Ker}(\Omega^p_X \to \Omega^p_Y)$ for $0 \leq p \leq n-1$ and $S^n = \Omega^n_X$. Then Thm. 1.4 (1) is equivalent to

$$\mathrm{Alb}(X, Y) \cong H_{\mathbb{Z}} \backslash H^{2n-1}(X, S)/H^{n-1}(X, \Omega^n_X)$$

where $H_{\mathbb{Z}} = H^{2n-1}_c(X - Y, \mathbb{Z}(n))/(\text{torsion part})$. As is shown in § 6, we have a commutative diagram with an isomorphism in the lower row

$$\begin{array}{ccc} H^{n-1}(X, \Omega^n_X) & = & H^{n-1}(X, \Omega^n_X) \\ \downarrow & & \downarrow \\ H^{2n-1}(X, S) & \cong & H^{2n-1}_c(X - Y, \mathbb{C}) \oplus H^{n-1}(X, \Omega^n_X / J\Omega^n_X). \end{array}$$

Thus (1) and (2) of Thm. 1.4 are deduced from each other.

**1.6.** As mentioned above, Thm. 1.4 shows that $\mathrm{Alb}(X,Y)$ is expressed as $H_{\mathbb{Z}}\backslash H_V/F^0$ where:

$H_{\mathbb{Z}} = H_c^{2n-1}(X-Y,\mathbb{Z}(n))/(\text{torsion part})$,

$H_V = H_{\mathbb{C}} \oplus H^{n-1}(X,\Omega_X^n/J\Omega_X^n) \cong H^{2n-1}(X,S)$
($H_{\mathbb{C}} = \mathbb{C}\otimes H_{\mathbb{Z}}$ and $S$ is as in 1.5 (d)),

$F^\bullet$ is the decreasing filtration on $H_V$ given by

$$F^{-1} = H_V, \quad F^0 = H^{n-1}(X,\Omega_X^n), \quad F^1 = 0.$$

Note that $H_V$ can be different from $H_{\mathbb{C}}$ here, and so $(H_{\mathbb{Z}}, H_V, F^\bullet)$ here need not be a Hodge structure. It is some kind of "mixed Hodge structure with additive part". This object $(H_{\mathbb{Z}}, H_V, F^\bullet)$ with a weight filtration, which we will denote by $H^{2n-1}(X,Y_-)(n)$ in section 6, belongs to a category $\mathcal{H}$ introduced in section 2 which contains the category of mixed Hodge structures but is bigger than it. In the proof of Thm. 1.4, it is essential to consider such object. This category $\mathcal{H}$ is related to the category of refined Hodge structures of Bloch-Srinivas [BlS] and to the category of formal Hodge structures of Barbieri-Viale [Ba]. In the proof of Thm. 1.4, we use the result of Barbieri-Viale in [Ba] on the Hodge theoretic description of the category of "1-motives with additive parts" over $\mathbb{C}$.

**1.7.** The theory of generalized Albanese variety in characteristic $p>0$ is given in [Ru3] basing on the duality theory [Ru2] of 1-motives with unipotent parts in characteristic $p>0$.

In characteristic $p>0$, the syntomic cohomology is an analogue of the Deligne cohomology. We expect that we can have presentations of the $p$-adic completion of $\mathrm{Alb}(X,Y)(k)$ ($k$ is the base field), which is similar to Thm. 1.4, by using crystalline cohomology theory and syntomic cohomology theory.

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# 2 Mixed Hodge structures with additive parts

**2.1.** For a proper smooth variety $X$ over $\mathbb{C}$ of dimension $n$ and for an effective divisor $Y$ on $X$, we will have in section 6 certain structures $H^1(X,Y_+)$ and $H^{2n-1}(X,Y_-)$ which are kinds of "mixed Hodge structures with additive parts". (These structures for the case $X$ is a curve are explained in 2.3 below.) The authors imagine that there is a nice definition of the category of "mixed Hodge structures with additive parts", which contains these $H^1(X,Y_+)$ and $H^{2n-1}(X,Y_-)$ as objects, but can not define it. Instead, we define a category $\mathcal{H}$ having these objects, which may be a very simple approximation of such nice category.

**2.2.** The category $\mathcal{H}$. An object of $\mathcal{H}$ is $H = (H_{\mathbb{Z}}, H_V, W_\bullet H_{\mathbb{Q}}, W_\bullet H_V, F^\bullet H_V, a, b)$, where $H_{\mathbb{Z}}$ is a finitely generated $\mathbb{Z}$-module, $H_V$ is a finite dimensional $\mathbb{C}$-vector space, $W_\bullet H_{\mathbb{Q}}$ is an increasing filtration on $H_{\mathbb{Q}} := \mathbb{Q}\otimes H_{\mathbb{Z}}$ (called weight filtration), $W_\bullet H_V$ is an increasing

filtration on $H_V$ (called weight filtration), $F^\bullet$ is a decreasing filtration on $H_V$ (called Hodge filtration), $a$ is a $\mathbb{C}$-linear map $H_\mathbb{C} := \mathbb{C} \otimes H_\mathbb{Z} \to H_V$ which sends $W_w H_\mathbb{C} := \mathbb{C} \otimes_\mathbb{Q} W_w H_\mathbb{Q}$ into $W_w H_V$ for any $w \in \mathbb{Z}$, and $b$ is a $\mathbb{C}$-linear map $H_V \to H_\mathbb{C}$ which sends $W_w H_V$ into $W_w H_\mathbb{C}$ for any $w \in \mathbb{Z}$ such that $b \circ a$ is the identity map of $H_\mathbb{C}$.

We sometimes denote an object $H$ of $\mathcal{H}$ simply as $(H_\mathbb{Z}, H_V)$.

The category of mixed Hodge structures is naturally embedded in $\mathcal{H}$ as a full subcategory, by putting $H_V = H_\mathbb{C}$.

The full subcategory of $\mathcal{H}$ consisting of all objects $H$ such that $H_\mathbb{Z}$ are torsion free is clearly self-dual.

This category $\mathcal{H}$ is a simpler version of the category of enriched Hodge structures in Bloch-Srinivas [BlS].

**2.3. Example.** Let $X$ be a proper smooth curve over $\mathbb{C}$ and let $Y$ be an effective divisor on $X$. Let $I$ be the ideal of $\mathcal{O}_X$ which defines $Y$, let $I_1$ be the ideal of $\mathcal{O}_X$ which defines the reduced part of $Y$, and let $J = II_1^{-1} \subset \mathcal{O}_X$.

We define objects $H^1(X, Y_+)$ and $H^1(X, Y_-)$ of $\mathcal{H}$.

First, we define $H = H^1(X, Y_+)$. Let

$$H_\mathbb{Z} = H^1(X - Y, \mathbb{Z}), \quad H_V = H^1(X, [\mathcal{O}_X \xrightarrow{d} I^{-1}\Omega_X^1]).$$

The map $a : H_\mathbb{C} \to H_V$ is

$$H^1(X - Y, \mathbb{C}) \cong H^1(X, [\mathcal{O}_X \to I_1^{-1}\Omega_X^1]) \longrightarrow H^1(X, [\mathcal{O}_X \to I^{-1}\Omega_X^1]).$$

The map $b : H_V \to H_\mathbb{C}$ is the composition

$$H^1(X, [\mathcal{O}_X \to I^{-1}\Omega_X^1]) \longrightarrow H^1(X, [J^{-1} \to I^{-1}\Omega_X^1]) \xleftarrow{\simeq} H^1(X, [\mathcal{O}_X \to I_1^{-1}\Omega_X])$$
$$\cong H^1(X - Y, \mathbb{C}).$$

The weight filtrations and the Hodge filtration are given by

$$W_2 H_\mathbb{Q} = H_\mathbb{Q}, \quad W_1 H_\mathbb{Q} = H^1(X, \mathbb{Q}), \quad W_0 H_\mathbb{Q} = 0,$$

$$W_2 H_V = H_V, \quad W_1 H_V = H^1(X, \mathbb{C}), \quad W_0 H_V = 0,$$

where $H^1(X, \mathbb{C})$ is embedded in $H_V$ via $a$, and

$$F^0 H_V = H_V, \quad F^1 H_V = H^1(X, \mathbb{C}), \quad F^2 H_\mathbb{C} = 0.$$

Next, we define $H = H^1(X, Y_-)$. Let

$$H_\mathbb{Z} = H_c^1(X - Y, \mathbb{Z}), \quad H_V = H^1(X, [I \xrightarrow{d} \Omega_X^1]).$$

The map $a : H_\mathbb{C} \to H_V$ is the composition

$$H_c^1(X - Y, \mathbb{C}) \cong H^1(X, [I_1 \to \Omega_X^1]) \xleftarrow{\simeq} H^1(X, [I \to J\Omega_X^1]) \longrightarrow H^1(X, [I \to \Omega_X^1]).$$

The map $b : H_V \to H_\mathbb{C}$ is

$$H^1(X, [I \to \Omega_X^1]) \longrightarrow H^1(X, [I_1 \to \Omega_X^1]) \cong H_c^1(X - Y, \mathbb{C}).$$

The weight filtrations and the Hodge filtration are given by

$$W_1 H_{\mathbb{Q}} = H_{\mathbb{Q}}, \quad W_0 H_Q = \mathrm{Ker}(H_{\mathbb{Q}} \to H^1(X, \mathbb{Q}), \quad W_{-1} H_{\mathbb{Q}} = 0,$$

$$W_1 H_V = H_V, \quad W_0 H_V = \mathrm{Ker}(H_V \to H^1(X, \mathbb{C})), \quad W_{-1} H_V = 0,$$

where $H^1(X, \mathbb{C})$ is regarded as quotient of $H_V$ via $b$, and

$$F^0 H_V = H_V, \quad F^1 H_V = \mathrm{Ker}(H_V \to H^1(X, \mathcal{O}_X)), \quad F^2 H_{\mathbb{C}} = 0.$$

Then we have exact sequences in $\mathcal{H}$

$$0 \longrightarrow H^1(X) \longrightarrow H^1(X, Y_+) \longrightarrow H^0(Y)(-1) \longrightarrow \mathbb{Z}(-1) \longrightarrow 0,$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow H^0(Y) \longrightarrow H^1(X, Y_-) \longrightarrow H^1(X) \longrightarrow 0.$$

Here for $r \in \mathbb{Z}$, $\mathbb{Z}(r)$ is the usual Hodge structure $\mathbb{Z}(r)$ regarded as an object of $\mathcal{H}$. $H^1(X)$ is also the usual Hodge structure of weight 1 associated to the first cohomology of $X$, regarded as an object of $\mathcal{H}$. Finally the object $H^0(Y)$ of $\mathcal{H}$ is defined as below, and $H^0(Y)(-1)$ is the $-1$ Tate twist.

The definition of $H = H^0(Y)$ is as follows. $H_{\mathbb{Z}} = H^0(Y, \mathbb{Z}) = \oplus_{y \in Y} \mathbb{Z}$. $H_V = H^0(Y, \mathcal{O}_Y)$. $a$ is the canonical map $H^0(Y, \mathbb{C}) \to H^0(Y, \mathcal{O}_Y)$. $b$ is the canonical map $H^0(Y, \mathcal{O}_Y) \to H^0(Y, \mathbb{C})$ given by $\mathcal{O}_Y \to \mathbb{C}$ which is $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}/m_y = \mathbb{C}$ at each $y \in Y$ ($m_y$ denotes the maximal ideal of $\mathcal{O}_{Y,y}$). The weight filtration and the Hodge filtration are given by

$$W_0 H = H, \quad W_{-1} H = 0,$$

$$F^0 H_V = H_V, \quad F^1 H_V = 0.$$

Note that $H_{\mathbb{C}} \to H_V$ can be like $\mathbb{C} \to \mathbb{C}[T]/(T^n)$, and need not be an isomorphism.

The evident self-duality $\underline{\mathrm{Hom}}(\ \ , \mathbb{Z})$ for torsion free objects in $\mathcal{H}$ induces

$$H^1(X, Y_-) \cong \underline{\mathrm{Hom}}(H^1(X, Y_+), \mathbb{Z})(-1).$$

# 3 1-motives with additive parts

In [La], Laumon formulated the notion "1-motive with additive part" over a field of characteristic 0. We review it assuming that the base field is algebraically closed for simplicity.

Fix an algebraically closed field $k$ of characteristic 0.

**3.1.** Let $\mathcal{A}b/k$ be the category of sheaves of abelian groups on the fppf-site of the category of affine schemes over $k$. Let $\mathcal{C}^{[-1,0]}(\mathcal{A}b/k)$ be the abelian category of complexes in $\mathcal{A}b/k$ concentrated in degrees $-1$ and 0.

A 1-*motive with additive part* over $k$ is an object of $\mathcal{C}^{[-1,0]}(\mathcal{A}b/k)$ of the form $[\mathcal{F} \to G]$, where $G$ is a commutative connected algebraic group over $k$ and $\mathcal{F} \cong \mathbb{Z}^t \oplus (\widehat{\mathbb{G}}_a)^s$ for some $t$ and $s$. (Cf. [La, Def. (5.1.1)].) Here $\mathbb{Z}$ is regarded as a constant sheaf and $\widehat{\mathbb{G}}_a$ denotes

the formal completion of the additive group $\mathbb{G}_a$ at 0. Recall that for any commutative ring $R$, $\widehat{\mathbb{G}}_a(R)$ is the subgroup of the additive group $R$ consisting of all nilpotent elements. We have $\mathcal{F} = \mathcal{F}_{\text{ét}} \oplus \mathcal{F}_{\text{inf}}$ where $\mathcal{F}_{\text{ét}}$ is the étale part of $\mathcal{F}$ which corresponds to $\mathbb{Z}^t$ in the above isomorphism and $\mathcal{F}_{\text{inf}}$ is the infinitesimal part of $\mathcal{F}$ which corresponds to $(\widehat{\mathbb{G}}_a)^s$.

We denote the category of 1-motives with additive parts over $k$ by $\mathcal{M}_1$.

**3.2.** The category $\mathcal{M}_1$ admits a notion of duality (called "Cartier duality"). Let $[\mathcal{F} \to G]$ be a 1-motive with additive part over $k$. Then we have the "Cartier dual" $[\mathcal{F}^* \to G^*]$ of $[\mathcal{F} \to G]$ which is an object of $\mathcal{M}_1$ obtained as follows. Let $0 \to L \to G \to A \to 0$ be the canonical decomposition of $G$ as an extension of an abelian variety $A$ by a commutative connected affine algebraic group $L$. Note that $L \cong (\mathbb{G}_m)^t \oplus (\mathbb{G}_a)^s$ for some $t$ and $s$. We have

$$\mathcal{F}^* = \underline{\mathrm{Hom}}_{\mathcal{A}b/k}(L, \mathbb{G}_m), \quad G^* = \underline{\mathrm{Ext}}^1_{\mathcal{C}^{[-1,0]}(\mathcal{A}b/k)}([\mathcal{F} \to A], \mathbb{G}_m)$$

and the homomorphism $\mathcal{F}^* \to G^*$ is the connecting homomorphism

$$\underline{\mathrm{Hom}}_{\mathcal{A}b/k}(L, \mathbb{G}_m) \longrightarrow \underline{\mathrm{Ext}}^1_{\mathcal{C}^{[-1,0]}(\mathcal{A}b/k)}([\mathcal{F} \to A], \mathbb{G}_m)$$

associated to the short exact sequence $0 \longrightarrow L \longrightarrow [\mathcal{F} \to G] \longrightarrow [\mathcal{F} \to A] \longrightarrow 0$ in $\mathcal{C}^{[-1,0]}(\mathcal{A}b/k)$. Since

$$\underline{\mathrm{Hom}}_{\mathcal{A}b/k}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}, \quad \underline{\mathrm{Hom}}_{\mathcal{A}b/k}(\mathbb{G}_a, \mathbb{G}_m) \cong \widehat{\mathbb{G}}_a,$$

we have $\mathcal{F}^* \simeq \mathbb{Z}^t \oplus (\widehat{\mathbb{G}}_a)^s$ for some $t$ and $s$. We have an exact sequence

$$0 \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{A}b/k}(\mathcal{F}, \mathbb{G}_m) \longrightarrow \underline{\mathrm{Ext}}^1_{\mathcal{C}^{[-1,0]}(\mathcal{A}b/k)}([\mathcal{F} \to A], \mathbb{G}_m) \longrightarrow \underline{\mathrm{Ext}}^1_{\mathcal{A}b/k}(A, \mathbb{G}_m) \longrightarrow 0,$$

$\underline{\mathrm{Ext}}^1_{\mathcal{A}b/k}(A, \mathbb{G}_m)$ is the dual abelian variety of $A$, and since

$$\underline{\mathrm{Hom}}_{\mathcal{A}b/k}(\mathbb{Z}, \mathbb{G}_m) \cong \mathbb{G}_m, \quad \underline{\mathrm{Hom}}_{\mathcal{A}b/k}(\widehat{\mathbb{G}}_a, \mathbb{G}_m) \cong \mathbb{G}_a,$$

$\underline{\mathrm{Hom}}_{\mathcal{A}b/k}(\mathcal{F}, \mathbb{G}_m) \cong (\mathbb{G}_m)^t \oplus (\mathbb{G}_a)^s$ for some $t$ and $s$. Hence $G^*$ is a commutative connected algebraic group over $k$. Thus $[\mathcal{F}^* \to G^*]$ is a 1-motive with additive part. The Cartier dual of $[\mathcal{F}^* \to G^*]$ is canonically isomorphic to $[\mathcal{F} \to G]$.

See [La, section 5] for details or [Ru, section 1] for a review.

**3.3.** Let $\mathcal{M}_{1,\{-1,-2\}}$ be the full subcategory of $\mathcal{M}_1$ consisting of all objects $[\mathcal{F} \to G]$ such that $\mathcal{F} = 0$.

Let $\mathcal{M}_{1,\{0,-1\}}$ be the full subcategory of $\mathcal{M}_1$ consisting of all objects $[\mathcal{F} \to G]$ such that $G$ is an abelian variety.

Then the self-duality of $\mathcal{M}_1$ in 3.2 induces an anti-equivalence between the categories $\mathcal{M}_{1,\{-1,-2\}}$ and $\mathcal{M}_{1,\{0,-1\}}$.

# 4 Equivalences of categories

In [Ba], Barbieri-Viale constructed a Hodge theoretic category and proved that in the case the base field is $\mathbb{C}$, the category $\mathcal{M}_1$ is equivalent to his Hodge theoretic category. Here we reformulate his equivalence in the style which is convenient for us, by using the category $\mathcal{H}$ from section 2.

**4.1.** The category $\mathcal{H}_1$. An object of $\mathcal{H}_1$ is an object $H$ of $\mathcal{H}$ endowed with a splitting of the weight filtration on $\mathrm{Ker}(H_V \to H_{\mathbb{C}})$ satisfying the following conditions (i)–(iv).

(i) $H_{\mathbb{Z}}$ is torsion free, $F^{-1}H_V = H_V$, $F^1 H_V = 0$, $W_0 H = H$, $W_{-3}H = 0$.

(ii) $\mathrm{gr}^W_{-1} H$ is a polarizable Hodge structure of weight $-1$. That is, $\mathrm{gr}^W_{-1} H_{\mathbb{C}} = \mathrm{gr}^W_{-1} H_V$ and $\mathrm{gr}^W_{-1} H_{\mathbb{Z}}$ with the Hodge filtration on $\mathrm{gr}^W_{-1} H_{\mathbb{C}}$ is a polarizable Hodge structure of weight $-1$.

(iii) $F^0 \mathrm{gr}^W_0 H_V = \mathrm{gr}^W_0 H_V$.

(iv) $F^0 W_{-2} H_V = 0$.

Morphisms of $\mathcal{H}_1$ are the evident ones.

The category $\mathcal{H}_1$ is self-dual by the functor $\underline{\mathrm{Hom}}(\ ,\mathbb{Z})(1)$.

**4.2.** For a subset $\Delta$ of $\{0,-1,-2\}$, let $\mathcal{H}_{1,\Delta}$ be the full subcategory of $\mathcal{H}_1$ consisting of all objects $H$ such that $\mathrm{gr}^W_w H = 0$ unless $w \in \Delta$.

The categories $\mathcal{H}_{1,\{-1,-2\}}$ and $\mathcal{H}_{1,\{0,-1\}}$ are important for us. These categories are in fact defined as full subcategories of $\mathcal{H}$ without reference to the splitting of the weight filtration on $\mathrm{Ker}(H_V \to H_{\mathbb{C}})$, for the weight filtrations on $\mathrm{Ker}(H_V \to H_{\mathbb{C}})$ of objects of these categories are pure.

Thus $\mathcal{H}_{1,\{-1,-2\}}$ is the full subcategory of $\mathcal{H}$ consisting of all objects $H$ satisfying the following conditions (i)–(iii).

(i) $H_{\mathbb{Z}}$ is torsion free, $F^{-1}H_V = H_V$, $F^1 H_V = 0$, $W_{-1} H = H$, $W_{-3}H = 0$.

(ii) $\mathrm{gr}^W_{-1} H$ is a polarizable Hodge structure of weight $-1$.

(iii) $F^0 W_{-2} H_V = 0$.

For example, the Tate twist $H^1(X, Y_-)(1)$ of the object $H^1(X, Y_-)$ of $\mathcal{H}$ in 2.3 belongs to $\mathcal{H}_{1,\{-1,-2\}}$.

Similarly, $\mathcal{H}_{1,\{0,-1\}}$ is the full subcategory of $\mathcal{H}$ consisting of all objects $H$ satisfying the following conditions (i)–(iii).

(i) $H_{\mathbb{Z}}$ is torsion free, $F^{-1}H_V = H_V$, $F^1 H_V = 0$, $W_0 H = H$, $W_{-2}H = 0$.

(ii) $\mathrm{gr}^W_{-1} H$ is a polarizable Hodge structure of weight $-1$.

(iii) $F^0 \mathrm{gr}^W_0 H_V = \mathrm{gr}^W_0 H_V$.

For example, the Tate twist $H^1(X, Y_+)(1)$ of the object $H^1(X, Y_+)$ of $\mathcal{H}$ in 2.3 belongs to $\mathcal{H}_{1,\{0,-1\}}$.

The self-duality $\underline{\mathrm{Hom}}(\ ,\mathbb{Z})(1)$ of $\mathcal{H}_1$ induces an anti-equivalence between the categories $\mathcal{H}_{1,\{-1,-2\}}$ and $\mathcal{H}_{1,\{0,-1\}}$.

**Theorem 4.3.** *(This is a reformulation of the equivalence of categories proved by Barbieri-Viale in [Ba].) We have an equivalence of categories $\mathcal{H}_1 \simeq \mathcal{M}_1$ which is compatible with the dualities, and which induces the equivalences*

$$\mathcal{H}_{1,\{-1,0\}} \simeq \mathcal{M}_{1,\{-1,0\}}, \quad \mathcal{H}_{1,\{-2,-1\}} \simeq \mathcal{M}_{1,\{-2,-1\}}.$$

The equivalence $\mathcal{H}_1 \simeq \mathcal{M}_1$ is given in 4.4 and 4.5 below.

**4.4.** First we define the functor $\mathcal{H}_1 \to \mathcal{M}_1$.

Let $H$ be an object of $\mathcal{H}_1$. The corresponding object $[\mathcal{F} \to G]$ of $\mathcal{M}_1$ is as follows.

$$\begin{aligned} G &= W_{-1}H_{\mathbb{Z}}\backslash W_{-1}H_V/F^0W_{-1}H_V. \\ \mathcal{F}_{\text{ét}} &= \mathrm{gr}_0^W(H_{\mathbb{Z}}). \\ \mathcal{F}_{\text{inf}} &= \text{the formal completion of } \mathrm{Ker}(\mathrm{gr}_0^W(H_V) \to \mathrm{gr}_0^W(H_{\mathbb{C}})). \end{aligned}$$

Here $\mathcal{F}_{\text{ét}}$ is the étale part of $\mathcal{F}$ and $\mathcal{F}_{\text{inf}}$ is the infinitesimal part of $\mathcal{F}$. The homomorphism $\mathcal{F} = \mathcal{F}_{\text{ét}} \oplus \mathcal{F}_{\text{inf}} \to G$ is given as follows.

The part $\mathcal{F}_{\text{ét}} \to G$: Let $x \in \mathcal{F}_{\text{ét}} = \mathrm{gr}_0^W H_{\mathbb{Z}}$. Since the sequence $0 \to W_{-1}H_{\mathbb{Z}} \to H_{\mathbb{Z}} \to \mathrm{gr}_0^W H_{\mathbb{Z}} \to 0$ is exact, we can lift $x$ to an element $y$ of $H_{\mathbb{Z}}$ and this lifting is unique modulo $W_{-1}H_{\mathbb{Z}}$. Since the sequence $0 \to F^0W_{-1}H_V \to F^0H_V \to F^0\mathrm{gr}_0^W H_V \to 0$ is exact, we can lift $x$ to an element $z$ of $F^0H_V$ and this lifting is unique modulo $F^0W_{-1}H_V$. Note that $y - z \in W_{-1}H_V$. We have a well-defined homomorphism

$$\mathcal{F}_{\text{ét}} = \mathrm{gr}_0^W H_{\mathbb{Z}} \longrightarrow W_{-1}H_{\mathbb{Z}}\backslash W_{-1}H_V/F^0W_{-1}H_V = G\ ;\ x \mapsto y - z.$$

The part $\mathcal{F}_{\text{inf}} \to G$: Note that $\mathrm{Hom}(\mathcal{F}_{\text{inf}}, G)$ is identified with $\mathrm{Hom}_{\mathbb{C}}(\mathrm{Lie}(\mathcal{F}_{\text{inf}}), \mathrm{Lie}(G))$. We give the corresponding homomorphism $\mathrm{Lie}(\mathcal{F}_{\text{inf}}) = \mathrm{Ker}(\mathrm{gr}_0^W(H_V) \to \mathrm{gr}_0^W(H_{\mathbb{C}})) \to \mathrm{Lie}(G) = W_{-1}H_V/F^0W_{-1}H_V$. Let $x \in \mathrm{Ker}(\mathrm{gr}_0^W(H_V) \to \mathrm{gr}_0^W(H_{\mathbb{C}}))$. The given splitting of the weight filtration on $\mathrm{Ker}(H_V \to H_{\mathbb{C}})$ sends $x$ to an element $y$ of $\mathrm{Ker}(H_V \to H_{\mathbb{C}})$. Since the sequence $0 \to F^0W_{-1}H_V \to F^0H_V \to F^0\mathrm{gr}_0^W H_V \to 0$ is exact, we can lift $x$ to an element $z$ of $F^0H_V$ and this lifting is unique modulo $F^0W_{-1}H_V$. Note that $y - z \in W_{-1}H_V$. We have a well-defined homomorphism

$$\mathrm{Ker}(\mathrm{gr}_0^W H_V \to \mathrm{gr}_0^W H_{\mathbb{C}}) \longrightarrow W_{-1}H_V/F^0W_{-1}H_V = \mathrm{Lie}(G)\ ;\ x \mapsto y - z.$$

**4.5.** We give the functor $\mathcal{M}_1 \to \mathcal{H}_1$.

Let $[\mathcal{F} \to G]$ be an object of $\mathcal{M}_1$. The corresponding object $H$ of $\mathcal{H}_1$ is as follows. Let $0 \to L \to G \to A \to 0$ be the exact sequence of commutative algebraic groups where $A$ is an abelian variety and $L$ is affine. Let $\mathcal{F}_{\text{ét}}$ be the étale part of $\mathcal{F}$ and let $\mathcal{F}_{\text{inf}}$ be the infinitesimal part of $\mathcal{F}$.

First, $H_{\mathbb{Z}}$ is the fiber product of $\mathcal{F}_{\text{ét}} \to G \leftarrow \mathrm{Lie}(G)$ where $\mathrm{Lie}(G) \to G$ is the exponential map so we have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & H_1(G,\mathbb{Z}) & \to & H_{\mathbb{Z}} & \to & \mathcal{F}_{\text{ét}} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H_1(G,\mathbb{Z}) & \to & \mathrm{Lie}(G) & \to & G & \to & 0. \end{array}$$

The weight filtration on $H_{\mathbb{Z}}$ is given as follows.

$$W_0H_{\mathbb{Z}} = H_{\mathbb{Z}}, \quad W_{-1}H_{\mathbb{Z}} = H_1(G,\mathbb{Z}),$$

$$W_{-2}H_{\mathbb{Z}} = H_1(L,\mathbb{Z}) = \mathrm{Ker}(H_1(G,\mathbb{Z}) \to H_1(A,\mathbb{Z})), \quad W_{-3}H_{\mathbb{Z}} = 0.$$

Next,

$$H_V = H_{\mathbb{C}} \oplus \mathrm{Lie}(L_a) \oplus \mathrm{Lie}(\mathcal{F}_{\text{inf}})$$

where $L_a$ is the additive part of $L$. The weight filtration on $H_V$ is as follows.

$$W_0 H_V = H_V, \quad W_{-1} H_V = H_1(G, \mathbb{C}) \oplus \mathrm{Lie}(L_a),$$

$$W_{-2} H_V = H_1(L, \mathbb{C}) \oplus \mathrm{Lie}(L_a), \quad W_{-3} H_V = 0.$$

The splitting of the weight filtration on $\mathrm{Ker}(H_V \to H_{\mathbb{C}}) = \mathrm{Lie}(L_a) \oplus \mathrm{Lie}(\mathcal{F}_{\mathrm{inf}})$ is defined to be this direct decomposition.

The Hodge filtration on $H_V$ is given as follows.

$$F^{-1} H_V = H_V, \quad F^1 H_V = 0,$$

$$F^0 H_V = \mathrm{Ker}(H_V \to \mathrm{Lie}(G))$$

where $H_V \to \mathrm{Lie}(G)$ is defined as follows. The part $H_{\mathbb{C}} \to \mathrm{Lie}(G)$ of it is the $\mathbb{C}$-linear map induced by the canonical map $H_{\mathbb{Z}} \to \mathrm{Lie}(G)$. The part $\mathrm{Lie}(L_a) \to \mathrm{Lie}(G)$ of it is the inclusion map. The part $\mathrm{Lie}(\mathcal{F}_{\mathrm{inf}}) \to \mathrm{Lie}(G)$ of it is the homomorphism induced by $\mathcal{F}_{\mathrm{inf}} \to G$. We have hence $H_V / F^0 H_V \cong \mathrm{Lie}(G)$.

It is easy to see that this functor $\mathcal{M}_1 \to \mathcal{H}_1$ is quasi-inverse to the functor $\mathcal{H}_1 \to \mathcal{M}_1$ in 4.4.

**4.6.** The induced functor $\mathcal{H}_{1,\{-1,-2\}} \xrightarrow{\simeq} \mathcal{M}_{1,\{-1,-2\}}$ is especially simple. It is

$$H \longmapsto [0 \to H_{\mathbb{Z}} \backslash H_V / F^0 H_V].$$

# 5 Generalized Albanese varieties

Let $k$ be an algebraically closed field of characteristic 0 and let $X$ be a proper smooth algebraic variety over $k$ of dimension $n$. We review generalized Albanese varieties $\mathrm{Alb}_{\mathcal{F}}(X)$ defined in [Ru][1]. For an effective divisor $Y$ on $X$, the generalized Albanese variety $\mathrm{Alb}(X, Y)$ of modulus $Y$ is a special case of $\mathrm{Alb}_{\mathcal{F}}(X)$.

The Albanese variety $\mathrm{Alb}(X)$ is defined by a universal mapping property for morphisms from $X$ to abelian varieties. Similarly, the generalized Albanese variety $\mathrm{Alb}(X, Y)$ of modulus $Y$ is characterized by a universal property for morphisms from $X - Y$ into commutative algebraic groups with "modulus" $\leq Y$. See Prop. 5.6.

**5.1.** Let $\underline{\mathrm{Div}}_X$ be the sheaf of abelian groups on $\mathcal{A}b/k$ defined as follows. For any commutative ring $R$ over $k$, $\underline{\mathrm{Div}}_X(R)$ is the group of all Cartier divisors on $X \otimes_k R$ generated locally on $\mathrm{Spec}(R)$ by effective Cartier divisors which are flat over $R$. Let $\underline{\mathrm{Pic}}_X$ be the Picard functor, and let $\underline{\mathrm{Pic}}_X^0 \subset \underline{\mathrm{Pic}}_X$ be the Picard variety of $X$. We have the class map $\underline{\mathrm{Div}}_X \to \underline{\mathrm{Pic}}_X$. Let $\underline{\mathrm{Div}}_X^0 \subset \underline{\mathrm{Div}}_X$ be the inverse image of $\underline{\mathrm{Pic}}_X^0$.

**5.2.** Let $\Lambda$ be the set of all subgroup sheaves $\mathcal{F}$ of $\underline{\mathrm{Div}}_X^0$ such that $\mathcal{F} \cong \mathbb{Z}^t \oplus (\widehat{\mathbb{G}}_a)^s$ for some $t$ and $s$. For $\mathcal{F} \in \Lambda$, we have an object $[\mathcal{F} \to \underline{\mathrm{Pic}}_X^0]$ of $\mathcal{M}_{1,\{0,-1\}}$. The generalized Albanese variety $\mathrm{Alb}_{\mathcal{F}}(X)$ is defined in [Ru] to be the Cartier dual of $[\mathcal{F} \to \underline{\mathrm{Pic}}_X^0]$. It is an object of $\mathcal{M}_{1,\{-1,-2\}}$ and hence is a commutative connected algebraic group over $k$.

If $\mathcal{F}, \mathcal{F}' \in \Lambda$ and $\mathcal{F} \subset \mathcal{F}'$, we have a canonical surjective homomorphism $\mathrm{Alb}_{\mathcal{F}'}(X) \to \mathrm{Alb}_{\mathcal{F}}(X)$. In the case $\mathcal{F} = 0$, $\mathrm{Alb}_{\mathcal{F}}(X) = \mathrm{Alb}(X)$.

[1] In [Ru], X was assumed to be projective. This assumption was used only for singular X, which is not our concern here. The construction of the $\mathrm{Alb}_{\mathcal{F}}(X)$ is valid in the same way for proper X.

**5.3.** Let $Y$ be an effective divisor of $X$. Then the generalized Albanese variety with modulus $Y$ is defined as $\mathrm{Alb}_{\mathcal{F}}(X)$ where $\mathcal{F} = \mathcal{F}_{X,Y} \in \Lambda$ is defined as follows. The étale part $\mathcal{F}_{\text{ét}}$ of $\mathcal{F}$ is the subgroup of $\underline{\mathrm{Div}}^0_X(k)$ consisting of all divisors whose support is contained in the support of $Y$. The infinitesimal part $\mathcal{F}_{\text{inf}}$ of $\mathcal{F}$ is as follows. Let $I$ be the ideal of $\mathcal{O}_X$ (though the notation $\mathcal{O}_X$ is often used in this paper for the sheaf of analytic functions, $\mathcal{O}_X$ here stands for the usual algebraic object on the Zariski site) defining $Y$, let $I_1$ be the ideal of $\mathcal{O}_X$ which defines the reduced part of $Y$, and let $J = II_1^{-1} \subset \mathcal{O}_X$. Then $\mathcal{F}_{\text{inf}}$ is the formal completion $\widehat{\mathbb{G}}_a \otimes_k H^0(X, J^{-1}/\mathcal{O}_X)$ of the finite dimensional $k$-vector space $H^0(X, J^{-1}/\mathcal{O}_X)$, which is embedded in $\underline{\mathrm{Div}}^0_X$ by the exponential map

$$\exp \;:\; \widehat{\mathbb{G}}_a \otimes_k H^0(X, J^{-1}/\mathcal{O}_X) \to \underline{\mathrm{Div}}^0_X.$$

If $Y'$ is an effective divisor on $X$ such that $Y' \geq Y$, then $\mathcal{F}_{X,Y} \subset \mathcal{F}_{X,Y'}$ and hence we have a canonical surjective homomorphism $\mathrm{Alb}(X, Y') \to \mathrm{Alb}(X, Y)$. In the case $Y = 0$, $\mathrm{Alb}(X, Y) = \mathrm{Alb}(X)$.

In the case $X$ is a curve, $\mathrm{Alb}(X, Y)$ coincides with the generalized Jacobian variety $J(X, Y)$ of $X$ with modulus $Y$ as is explained in [Ru, Exm. 2.34].

**5.4.** As in [Ru], for $\mathcal{F} \in \Lambda$, we have a rational map

$$\alpha_{\mathcal{F}} : X \to \mathrm{Alb}_{\mathcal{F}}(X)$$

which is canonically defined up to translation by a $k$-rational point of $\mathrm{Alb}_{\mathcal{F}}(X)$. If $\mathcal{F}' \in \Lambda$ and $\mathcal{F} \subset \mathcal{F}'$, then $\alpha_{\mathcal{F}}$ and $\alpha_{\mathcal{F}'}$ are compatible via the canonical surjection $\mathrm{Alb}_{\mathcal{F}'}(X) \to \mathrm{Alb}_{\mathcal{F}}(X)$.

For an effective divisor $Y$ on $X$, we denote the rational map $\alpha_{\mathcal{F}_{X,Y}}$ simply by $\alpha_{X,Y}$. In Prop. 5.6 (2) below, we give a universal property of $\alpha_{X,Y} : X \to \mathrm{Alb}(X, Y)$ concerning rational maps from $X$ to commutative algebraic groups. This property follows from a general universal property of $\alpha_{\mathcal{F}} : X \to \mathrm{Alb}_{\mathcal{F}}(X)$ obtained in [Ru], as is shown in 5.7 below.

**5.5.** Let $G$ be a commutative connected algebraic group over $k$ and let $\varphi : X \to G$ be a rational map. We define an effective divisor $\mathrm{mod}(\varphi)$ on $X$ which we call the modulus of $\varphi$.

We treat $X$ as a scheme. This divisor $\mathrm{mod}(\varphi)$ is written in the form $\sum_v \mathrm{mod}_v(\varphi)v$, where $v$ ranges over all points of $X$ of codimension one and $\mathrm{mod}_v(\varphi)$ is a non-negative integer defined as follows.

Let $0 \to L \to G \to A \to 0$ be the canonical decomposition of $G$ and take an isomorphism

(1) $\quad L_a \cong (\mathbb{G}_a)^s$

where $L_a$ is the additive part of $L$.

Let $K$ be the function field of $X$, and regard $\varphi$ as an element of $G(K)$. Since the local ring $\mathcal{O}_{X,v}$ of $X$ at $v$ is a discrete valuation ring and since $A$ is proper, we have $A(O_{X,v}) = A(K)$. By the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & L(O_{X,v}) & \to & G(\mathcal{O}_{X,v}) & \to & A(O_{X,v}) & \to & 0 \\ & & \cap & & \cap & & \| & & \\ 0 & \to & L(K) & \to & G(K) & \to & A(K) & \to & 0, \end{array}$$

we have $G(K) = L(K)G(O_{X,v})$. Write $\varphi \in G(K)$ as

(2) $\quad \varphi = lg \quad$ with $l \in L(K)$ and $g \in G(O_{X,v})$,

Let $(l_j)_{1\le j\le s}$ be the image of $l$ in $(\mathbb{G}_a)^s(K)$.

If $\varphi$ belongs to $G(O_{X,v})$, we define $\mathrm{mod}_v(\varphi) = 0$. Assume $\varphi$ does not belong to $G(O_{X,v})$. Then we define

$$\mathrm{mod}_v(\varphi) = 1 + \max(\{-\mathrm{ord}_v(l_j) \mid 1 \le j \le s\} \cup \{0\}).$$

This integer $\mathrm{mod}_v(\varphi)$ is independent of the choice of the isomorphism (1) and of the choice of the presentation (2) of $\varphi$.

For example, if $G = \mathbb{G}_m$, $\mathrm{mod}_v(\varphi)$ is 0 if the element $\varphi$ of $G(K) = K^\times$ belongs to $\mathcal{O}_{X,v}^\times$, and is 1 otherwise. If $G = \mathbb{G}_a$, $\mathrm{mod}_v(\varphi)$ is 0 if the element $\varphi$ of $G(K) = K$ belongs to $\mathcal{O}_{X,v}$, and is $m+1$ if $\varphi$ has a pole of order $m \ge 1$ at $v$.

**Proposition 5.6.** *Let $G$ be a commutative connected algebraic group over $k$ and let $\varphi : X \to G$ be a rational map.*

(1) *For a dense open set $U$ of $X$, $\varphi$ induces a morphism $U \to G$ (not only a rational map) if and only if the support of* $\mathrm{mod}(\varphi)$ *does not meet $U$.*

(2) *Let $Y$ be an effective divisor on $X$. Then the following two conditions (i) and (ii) are equivalent.*

(i) *There is a homomorphism $h : \mathrm{Alb}(X,Y) \to G$ such that $\varphi$ coincides with $h \circ \alpha_{X,Y}$ modulo a translation by $G(k)$.*

(ii) $\quad \mathrm{mod}(\varphi) \le Y$.

*Furthermore, if these equivalent conditions are satisfied, such homomorphism $h$ is unique.*

It is easy to prove (1). The proof of (2) is given in 5.8 below after we review results on $\mathrm{Alb}_{\mathcal{F}}(X)$ from [Ru].

**5.7.** We review a general universal property of $\mathrm{Alb}_{\mathcal{F}}(X)$ proved in [Ru] concerning rational maps from $X$ into commutative algebraic groups.

Let $\varphi : X \to G$ be a rational map into a commutative connected algebraic group $G$, and let $L$ be the canonical connected affine subgroup such that the quotient $G/L$ is an abelian variety. One observes that $\varphi$ induces a natural transformation $\tau_\varphi : L^\vee \to \underline{\mathrm{Div}}^0_X$ (see [Ru, section 2.2]), where $L^\vee = \underline{\mathrm{Hom}}_{\mathcal{A}b/k}(L, \mathbb{G}_m)$ is the Cartier dual of $L$. It is shown in [Ru, section 2.3] that if $\mathcal{F} \in \Lambda$, there is a rational map $\alpha_{\mathcal{F}} : X \to \mathrm{Alb}_{\mathcal{F}}(X)$ for which the corresponding homomorphism $\tau_{\alpha_{\mathcal{F}}} : \mathcal{F} \to \underline{\mathrm{Div}}^0_X$ coincides with the inclusion map, and such rational map $\alpha_{\mathcal{F}}$ is unique up to translation by a $k$-rational point of $\mathrm{Alb}_{\mathcal{F}}(X)$. For a rational map $\varphi : X \to G$ into a commutative connected algebraic group $G$ and for $\mathcal{F} \in \Lambda$, there is a homomorphism $h : \mathrm{Alb}_{\mathcal{F}}(X) \to G$ such that $f$ coincides with $h \circ \alpha_{\mathcal{F}}$ up to a translation by $G(k)$ if and only if the image of the homomorphism $\tau_\varphi : L^\vee \to \underline{\mathrm{Div}}^0_X$ is contained in $\mathcal{F}$. Furthermore, if such $h$ exists, it is unique.

Moreover, any rational map $\varphi : X \to G$ into a commutative connected algebraic group $G$ coincides with $h \circ \alpha_{\mathcal{F}}$ up to a translation by $G(k)$ for some $\mathcal{F} \in \Lambda$ and for some homomorphism $h : \mathrm{Alb}_{\mathcal{F}}(X) \to G$. This is because there is always some $\mathcal{F} \in \Lambda$ which contains the image of $L^\vee \to \underline{\mathrm{Div}}^0_X$.

**5.8.** We prove Prop. 5.6. By 5.7 we find that condition (i) of 5.6 (2) is equivalent to

(i') $\operatorname{im}(\tau_\varphi) \subset \mathcal{F}_{X,Y}$.

Write

$$Y = \sum_v e_v v$$

where $v$ ranges over all points of $X$ of codimension one and $e_v \in \mathbb{N}$. Condition (ii) of 5.6 (2) is expressed as

(ii') $\operatorname{mod}_v(\varphi) \leq e_v$ for all points $v$ of codimension one in $X$.

Fix an isomorphism $L \cong (\mathbb{G}_m)^t \times (\mathbb{G}_a)^s$. For each point $v$ of $X$ of codimension one, take a presentation $\varphi = lg$ as in (2) in 5.5, let $(l'_{v,j})_{1\leq j\leq t}$ be the image of $l$ in $(\mathbb{G}_m)^t(K) = (K^\times)^t$, and as in 5.5, let $(l_{v,j})_{1\leq j\leq s}$ be the image of $l$ in $(\mathbb{G}_a)^s(K) = K^s$. Note that

(a) $\varphi \in G(\mathcal{O}_{X,v})$ if and only if $l'_{v,j} \in \mathcal{O}_{X,v}^\times$ for $1 \leq j \leq t$ and $l_{v,j} \in \mathcal{O}_{X,v}$ for $1 \leq j \leq s$.

By construction of the transformation $\tau_\varphi$ in [Ru, section 2.2], we have the following (b) and (c).

(b) The étale part of $\tau_\varphi$

$$\tau_{\varphi,\text{ét}} : \mathbb{Z}^t \to \underline{\mathrm{Div}}^0_X(k)$$

sends the $j$-th base of $\mathbb{Z}^t$ $(1 \leq j \leq t)$ to the divisor $\sum_v \operatorname{ord}_v(l'_{v,j})v$.

(c) The infinitesimal part of $\tau_\varphi$

$$\tau_{\varphi,\inf} : (\widehat{\mathbb{G}}_a)^s \to \underline{\mathrm{Div}}^0_X$$

has the form

$$(a_j)_{1\leq j\leq s} \mapsto \exp\left(\sum_{j=1}^s a_j f_j\right)$$

for some $f_j \in \Gamma(X, K/\mathcal{O}_X) = \operatorname{Lie}(\underline{\mathrm{Div}}^0_X)$ $(1 \leq j \leq s)$ such that for any point $v$ of $X$ of codimension one, the stalk of $f_j$ at $v$ coincides with $l_{v,j} \mod \mathcal{O}_{X,v}$.

Condition (i') is equivalent to the condition that the following (i'$_{\text{ét}}$) and (i'$_{\inf}$) are satisfied.

(i'$_{\text{ét}}$) The image of $\tau_{\varphi,\text{ét}}$ is contained in the étale part of $\mathcal{F}_{X,Y}$.

(i'$_{\inf}$) The image of $\tau_{\varphi,\inf}$ is contained in the infinitesimal part of $\mathcal{F}_{X,Y}$.

By the above (b), (i'$_{\text{ét}}$) is equivalent to the condition that the following (i'$_{\text{ét},v}$) is satisfied for any point $v$ of $X$ of codimension one.

(i'$_{\text{ét},v}$) If $e_v = 0$, then $l'_{v,j} \in \mathcal{O}_{X,v}^\times$ for $1 \leq j \leq t$.

On the other hand, by the above (c), (i'$_{\inf}$) is equivalent to

$$f_j \in \Gamma(X, J^{-1}/\mathcal{O}_X) \text{ for } 1 \leq j \leq s,$$

and hence equivalent to the condition that the following (i'$_{\inf,v}$) is satisfied for any point $v$ of $X$ of codimension one.

(i'$_{\inf,v}$) If $e_v = 0$, then $l_{v,j} \in \mathcal{O}_{X,v}$ for $1 \leq j \leq s$.
If $e_v \geq 1$, then $\operatorname{ord}_v(l_{v,j}) \geq 1 - e_v$ for $1 \leq j \leq s$.

By the above (a), for each $v$, (i'$_{\text{ét},v}$) and (i'$_{\inf,v}$) are satisfied if and only if $\operatorname{mod}_v(\varphi) \leq e_v$. $\square$

**Corollary 5.9.** *For any $\mathcal{F} \in \Lambda$, there exists an effective divisor $Y$ such that $\mathcal{F} \subset \mathcal{F}_{X,Y}$.*

*Proof.* Let $Y = \operatorname{mod}(\alpha_\mathcal{F})$ be the modulus of the rational map $\alpha_\mathcal{F} : X \to \operatorname{Alb}_\mathcal{F}(X)$ associated to $\mathcal{F} \in \Lambda$. Then $\mathcal{F} \subset \mathcal{F}_{X,Y}$. $\square$

# 6 Proof of Theorem 1.4

We prove Thm. 1.4. Let $X$ be a proper smooth algebraic variety over $\mathbb{C}$ of dimension $n$, and let $Y$ be an effective divisor on $X$. Let $I$ be the ideal of $\mathcal{O}_X$ which defines $Y$, let $I_1$ be the ideal of $\mathcal{O}_X$ which defines the reduced part of $Y$, and let $J = II_1^{-1} \subset \mathcal{O}_X$.

**6.1.** Let $H^1(X, Y_+)(1)$ be the object of $\mathcal{H}_{1,\{0,-1\}}$ corresponding to the object $[\mathcal{F}_{X,Y} \to \mathrm{Pic}^0(X)]$ of $\mathcal{M}_{1,\{0,-1\}}$ in the equivalence of categories 4.3. Let $H^{2n-1}(X, Y_-)(n)$ be the object of $\mathcal{H}_{1,\{-1,-2\}}$ corresponding to the object $\mathrm{Alb}(X, Y)$ of $\mathcal{M}_{1,\{-1,-2\}}$.

Since the equivalence of categories in 4.3 is compatible with dualities, we have

$$H^{2n-1}(X, Y_-)(n) \cong \underline{\mathrm{Hom}}(H^1(X, Y_+)(1), \mathbb{Z})(1). \tag{5}$$

We prove Thm. 1.4 in the following way. First in 6.3, we give an explicit description of $H^1(X, Y_+)(1)$. From this, by (5), we can obtain an explicit description of $H^{2n-1}(X, Y_-)(n)$ as in 6.4. Since $\mathrm{Alb}(X, Y)$ corresponds to $H^{2n-1}(X, Y_-)(n)$ in the equivalence of categories $\mathcal{H}_{1,\{-1,-2\}} \simeq \mathcal{M}_{1,\{-1,-2\}}$, we can obtain from 6.4 the explicit descriptions of $\mathrm{Alb}(X, Y)$ stated in Thm. 1.4.

We define objects $H^1(X, Y_+)$ and $H^{2n-1}(X, Y_-)$ of $\mathcal{H}$ as follows: $H^1(X, Y_+)$ is the Tate twist $(H^1(X, Y_+)(1))(-1)$ of $H^1(X, Y_+)(1)$, and $H^{2n-1}(X, Y_-)$ is the Tate twist $(H^{2n-1}(X, Y_-)(n))(-n)$ of $H^{2n-1}(X, Y_+)(n)$. These are natural generalizations of the objects of $\mathcal{H}$ for the curve case considered in 2.3.

**6.2.** We define canonical $\mathbb{C}$-linear maps

$$H^1(X - Y, \mathbb{C}) \longrightarrow H^1(X, \mathcal{O}_X), \tag{6}$$

$$H^{n-1}(X, \Omega^n_X) \longrightarrow H^{2n-1}_c(X - Y, \mathbb{C}) \tag{7}$$

First assume that $Y$ is with normal crossings. Then by [De], we have canonical isomorphisms

$$H^m(X - Y, \mathbb{C}) \cong H^m(X, \Omega^\bullet_X(\log(Y))), \quad H^m_c(X - Y, \mathbb{C}) \cong H^m(X, \Omega^\bullet_X(-\log(Y)))$$

for $m \in \mathbb{Z}$, where $\Omega^p_X(\log(Y))$ is the sheaf of differential $p$-forms with log poles along $Y$, and $\Omega^p_X(-\log(Y)) = I_1\Omega^p_X(\log(Y))$. Since $\mathcal{O}_X = \Omega^0_X(\log(Y))$ and $\Omega^n_X = \Omega^n_X(-\log(Y))$, we have canonical maps of complexes $\Omega^\bullet_X(\log(Y)) \to \mathcal{O}_X$ and $\Omega^n_X[-n] \to \Omega^\bullet_X(-\log(Y))$. These maps induces the maps (6) and (7) in the case $Y$ is with normal crossings, respectively.

In general, take a birational morphism $X' \to X$ of proper smooth algebraic varieties over $\mathbb{C}$ such that the inverse image $Y'$ of $Y$ on $X'$ is with normal crossings. Then we have maps

$$H^{n-1}(X, \Omega^n_X) \longrightarrow H^{n-1}(X', \Omega^n_{X'}) \longrightarrow H^{2n-1}_c(X' - Y', \mathbb{C}) = H^{2n-1}_c(X - Y, \mathbb{C})$$

where the second arrow is the map (7) for $X'$, and the composition $H^{n-1}(X, \Omega^n_X) \to H^{2n-1}_c(X - Y, \mathbb{C})$ is independent of the choice of $X' \to X$. The $\mathbb{C}$-linear dual of (7) with respect to the Poincaré duality and Serre duality gives the map (6). The map (6) is also obtained as the composition

$$H^1(X - Y, \mathbb{C}) = H^1(X' - Y', \mathbb{C}) \longrightarrow H^1(X', \mathcal{O}_{X'}) \xleftarrow{\simeq} H^1(X, \mathcal{O}_X).$$

**6.3.** Let $H = H^1(X, Y_+)(1)$, the object of $\mathcal{H}_{1,\{0,-1\}}$ corresponding to the object $[\mathcal{F}_{X,Y} \to \mathrm{Pic}^0(X)]$ of $\mathcal{M}_{1,\{0,-1\}}$. We describe $H$. By [BaS] Theorem 4.7 which treats the case $Y$ has no multiplicity, we can identify $H_{\mathbb{Z}}$ with $H^1(X-Y, \mathbb{Z}(1))$ and identify the map $H_{\mathbb{C}} \to \mathrm{Lie}(\mathrm{Pic}^0(X)) = H^1(X, \mathcal{O}_X)$ with the map (6) in 6.2. We have $H_V = H_{\mathbb{C}} \oplus H^0(X, J^{-1}/\mathcal{O}_X)$, the maps $a : H_{\mathbb{C}} \to H_V$ and $b : H_V \to H_{\mathbb{C}}$ are the evident ones, the weight filtration is given by $W_0 H = H$, $W_{-2}H = 0$,

$$W_{-1}H_{\mathbb{Q}} = H^1(X, \mathbb{Q}(1)), \quad W_{-1}H_V = H^1(X, \mathbb{C}),$$

and the Hodge filtration is given by $F^{-1}H_V = H_V$, $F^1 H_V = 0$, and

$$F^0 H_V = \mathrm{Ker}(H^1(X-Y, \mathbb{C}) \oplus H^0(J^{-1}/\mathcal{O}_X) \to H^1(X, \mathcal{O}_X))$$

where the map $H^0(J^{-1}/\mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ is the connecting map of the exact sequence $0 \to \mathcal{O}_X \to J^{-1} \to J^{-1}/\mathcal{O}_X \to 0$.

**6.4.** Let $H = H^{2n-1}(X, Y_-)(n)$, the object of $\mathcal{H}_{1,\{-1,-2\}}$ corresponding to the object $\mathrm{Alb}(X, Y)$ of $\mathcal{M}_{1,\{-1,-2\}}$. By (5) in 6.1, we obtain the following description of $H$ from the description of $H^1(X, Y_+)(1)$ in 6.3.

$$H_{\mathbb{Z}} = H_c^{2n-1}(X-Y, \mathbb{Z})/(\text{torsion}), \quad H_V = H_{\mathbb{C}} \oplus H^{n-1}(X, \Omega_X^n/J\Omega_X^n),$$

the maps $a : H_{\mathbb{C}} \to H_V$ and $b : H_V \to H_{\mathbb{C}}$ are the evident ones, the weight filtration is given by $W_{-1}H = H$, $W_{-3}H = 0$,

$$W_{-2}H_{\mathbb{Q}} = \mathrm{Ker}(H_{\mathbb{Q}} \to H^{2n-1}(X, \mathbb{Q}(n))), \quad W_{-2}H_V = \mathrm{Ker}(H_V \to H^{2n-1}(X, \mathbb{C})),$$

and the Hodge filtration is given by $F^{-1}H_V = H_V$, $F^1 H_V = 0$, and

$$F^0 H_V = \mathrm{Image}(H^{n-1}(X, \Omega_X^n) \longrightarrow H_c^{2n-1}(X-Y, \mathbb{C}) \oplus H^{n-1}(X, \Omega_X^n/J\Omega_X^n))$$

where the map $H^{n-1}(X, \Omega_X^n) \to H_c^{2n-1}(X-Y, \mathbb{C})$ is (7) in 6.2 and the map $H^{n-1}(X, \Omega_X^n) \to H^{n-1}(X, \Omega_X^n/J\Omega_X^n)$ is the evident one.

**6.5.** We prove Thm. 1.4 (2). Let $H = H^{2n-1}(X, Y_-)(n)$. Then

$$\mathrm{Alb}(X, Y) = H_{\mathbb{Z}} \backslash H_V / F^0 H_V$$

by 4.6. Hence the description of $H^{2n-1}(X, Y_-)(n)$ in 6.4 proves Thm. 1.4 (2).

**6.6.** As a preparation for the proof of Thm. 1.4 (1), we review a kind of Serre-duality obtained in Appendix by Deligne in the book [Ha].

Let $S$ be a proper scheme over a field $k$, let $C$ be a closed subscheme of $S$, let $U = S - C$, and let $I_C$ be the ideal of $\mathcal{O}_S$ which defines $C$. Assume $U$ is smooth over $k$ and purely of dimension $n$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_S$-module. Then for any $p \in \mathbb{Z}$, we have a canonical isomorphism

$$H^p(U, R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_U, \Omega_U^n)) \cong \varinjlim_m \mathrm{Hom}_k(H^{n-p}(X, I_C^m \mathcal{F}), k).$$

In the case $C$ is empty and $\mathcal{F}$ is locally free, this is the usual Serre duality

$$H^p(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \Omega_X^n)) \cong \mathrm{Hom}_k(H^{n-p}(X, \mathcal{F}), k).$$

**6.7.** We start the proof of Thm. 1.4 (1).

Let $C_Y$ be the subcomplex of $\Omega_X^\bullet$ defined as

$$C_Y^p = \ker(\Omega_X^p \to \Omega_Y^p) \ \text{ for } 0 \le p \le n-1, \quad C_Y^n = J\Omega_X^n.$$

**Proposition 6.8.** *For $p = 2n, 2n-1$, the maps $H_c^p(X - Y, \mathbb{C}) \to H^p(X, C_Y)$ induced by the homomorphism $j_!\mathbb{C} \to C_Y$ are isomorphisms.*

**6.9.** We prove Prop. 6.8 in the case $Y = Y_1$. We have an exact sequence of complexes

$$0 \longrightarrow C_{Y_1} \longrightarrow \Omega_X^\bullet \longrightarrow \Omega_{Y_1}^{\le n-1} \longrightarrow 0.$$

Since the support of $\Omega_{Y_1}^{\le n-1}$ is of dimension $\le n-1$ and since $\Omega_{Y_1}^{\le n-1}$ has only terms of degree $\le n-1$, we have $H^p(X, \Omega_{Y_1}^{\le n-1}) = 0$ for $p \ge 2n-1$. Hence

$$H^{2n}(X, C_{Y_1}) \cong H^{2n}(X, \Omega_X^\bullet) \cong H^{2n}(X, \mathbb{C}) \cong H_c^{2n}(X - Y, \mathbb{C}).$$

The above exact sequence of complexes induces the lower row of the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
H^{2n-2}(X,\mathbb{C}) & \to & H^{2n-2}(Y_1,\mathbb{C}) & \to & H_c^{2n-1}(X-Y_1,\mathbb{C}) & \to & H^{2n-1}(X,\mathbb{C}) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
H^{2n-2}(X,\Omega_X^\bullet) & \to & H^{2n-2}(Y_1,\Omega_{Y_1}^{\le n-1}) & \to & H^{2n-1}(X,C_{Y_1}) & \to & H^{2n-1}(X,\Omega_X^\bullet) & \to 0.
\end{array}$$

The vertical arrows except possibly the map $H_c^{2n-1}(X - Y_1, \mathbb{C}) \to H^{2n-1}(X, C_{Y_1})$ are isomorphisms. Hence the last map is also an isomorphism.

**Lemma 6.10.** *Let $Y'$ and $Y''$ be effective divisors on $X$ whose supports coincide with $Y_1$ and assume $Y' \ge Y''$. Then the canonical map $H^{2n-1}(X, C_{Y'}) \to H^{2n-1}(X, C_{Y''})$ is surjective and the canonical map $H^{2n}(X, C_{Y'}) \to H^{2n}(X, C_{Y''})$ is an isomorphism.*

*Proof.* Let $N = C_{Y''}/C_{Y'}$. We have

$$N^p = \operatorname{Ker}(\Omega_{Y'}^p \to \Omega_{Y''}^p) \quad \text{for } 0 \le p \le n-1, \quad N^n = J''\Omega_X^n / J'\Omega_X^n.$$

Here, $J' = I'I_1^{-1}$, $J'' = I''I_1^{-1}$ with $I'$ (resp. $I''$) the ideal of $\mathcal{O}_X$ which defines $Y'$ (resp. $Y''$). Since the support of $N$ is of dimension $\le n-1$ and $N$ has only terms of degree $\le n$, we have $H^{2n}(X, N) = 0$. Hence it is sufficient to prove $H^{2n-1}(X, N) = 0$.

Let $\Sigma$ be the set of all singular points of $Y_1$. Then $\Sigma$ is of dimension $\le n-2$. Let $\Omega_X^\bullet(\log(Y_1))$ be the de Rham complex on $X - \Sigma$ with log poles along $Y_1 - \Sigma$. Then as is easily seen, the restriction of $C_Y$ to $X - \Sigma$ coincides with $I\Omega_X^\bullet(\log(Y_1))$. Let $I_\Sigma$ be the ideal of $\mathcal{O}_X$ defining $\Sigma$ (here $\Sigma$ is endowed with the reduced structure). For $k \ge 0$, let $N_k$ be the subcomplex of $N$ defined by $N_k^p = I_\Sigma^{\max(k-p,0)} N^p$. In particular, $N_0 = N$. Then if $k \ge j \ge 0$, since the support of $N_j/N_k$ is of dimension $\le n-2$ and $N_j/N_k$ has only terms of degree $\le n$, we have $H^{2n-1}(X, N_j/N_k) = 0$. Hence $H^{2n-1}(X, N_k) \to H^{2n-1}(X, N_j)$ is surjective. Applying 6.6 for $S = X$ and $C = \Sigma$ yields that $\varprojlim_k H^{2n-1}(X, N_k)$ is the dual vector space of $H^0(X - \Sigma, [(J')^{-1}/(J'')^{-1} \xrightarrow{d} (J')^{-1}\Omega_X(\log Y_1)/(J'')^{-1}\Omega_X(\log Y_1)])$. Since $d : (J')^{-1}/(J'')^{-1} \to (J')^{-1}\Omega_X(\log Y_1)/(J'')^{-1}\Omega_X(\log Y_1)$ is injective, the last cohomology group is 0. Hence $H^{2n-1}(X, N_k) = 0$ for all $k \ge 0$. In particular, $H^{2n-1}(X, N) = 0$. $\square$

**6.11.** We prove Prop. 6.8 in general. By Lemma 6.10, the map $\varprojlim_{Y'} H^{2n-1}(X, C_{Y'}) \to H^{2n-1}(X, C_Y)$ is surjective, where $Y'$ ranges over all effective divisors on $X$ whose supports coincide with $Y_1$. By 6.6 which we apply by taking $S = X$ and $C = Y$, we have that $\varprojlim_{Y'} H^{2n-1}(X, C_{Y'})$ is the dual vector space of $H^1((X-Y)_{\mathrm{zar}}, \Omega^\bullet_{X-Y,\mathrm{alg}})$ where zar means Zariski topology and alg means the algebraic version. But $H^1((X-Y)_{\mathrm{zar}}, \Omega^\bullet_{X-Y,\mathrm{alg}}) \simeq H^1(X, \mathbb{C})$. This proves $\varprojlim_{Y'} H^{2n-1}(X, C_{Y'}) \cong H^{2n-1}_c(X - Y, \mathbb{C})$. Hence the map $H^{2n-1}_c(X-Y, \mathbb{C}) \to H^{2n-1}(X, C_Y)$ is surjective. Since the composition $H^{2n-1}_c(X-Y, \mathbb{C}) \to H^{2n-1}(X, C_Y) \to H^{2n-1}(X, C_{Y_1}) \cong H^{2n-1}_c(X-Y, \mathbb{C})$ is the identity map, the map $H^{2n-1}_c(X-Y, \mathbb{C}) \to H^{2n-1}(X, C_Y)$ is an isomorphism.

**6.12.** We prove (1) of Thm. 1.4. Let $S_Y = \mathrm{Ker}(\Omega^\bullet_X \to \Omega^{\leq n-1}_Y)$. Then $C_Y \subset S_Y \subset C_{Y_1}$. We have an exact sequence of complexes

$$0 \longrightarrow C_Y \longrightarrow S_Y \longrightarrow \Omega^n_X / J\Omega^n_X[-n] \longrightarrow 0.$$

Hence we have an exact sequence

$$H^{2n-1}(X, C_Y) \to H^{2n-1}(X, S_Y) \to H^{n-1}(X, \Omega^n_X / J\Omega^n_X) \to H^{2n}(X, C_Y) \to H^{2n}(X, S_Y).$$

Note that for $p = 2n, 2n-1$, the compositions

$$H^p(X, C_Y) \longrightarrow H^p(X, S_Y) \longrightarrow H^p(X, C_{Y_1})$$

are isomorphisms by Prop. 6.8. Hence by Prop. 6.8, we have an isomorphism

$$H^{2n-1}(X, S_Y) \cong H^{2n-1}_c(X-Y, \mathbb{C}) \oplus H^{n-1}(X, \Omega^n_X / J\Omega^n_X)$$

which is compatible with the maps from $H^{n-1}(X, \Omega^n_X)$. Hence (1) of Thm. 1.4 follows from (2) of Thm. 1.4.

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